Relativistic EPR Correlation and Bell's Inequality

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Introduction

- Bohm's version of the EPR experiment in the framework of relativistic quantum theory
- The effect of the observer's motion on the EPR correlation
- Group-theoretical approach
- Spin is entangled with momentum by the Wigner rotation

Massive Particle

- Two massive particles in the spin-singlet state
- Moving in opposite directions with definite momenta
- Two observers moving in the same direction at the same velocity with respect to the laboratory frame
- The two observes measure the spin in the same direction in their common rest frame
- By applying the Wigner rotation, the spins are not perfectly anti-correlated
- Seen from the moving observers, the anti-correlation in the same direction decreases unlike the non-relativistic case
- The perfect anti-correlation in every direction is not maintained in all inertial frames.
- The special directions where the perfect anti-correlation is not maintained are specified by the motion of the observers and particles.
- The motion of the observers decreases the degree of violation of Bell’s Inequality if the direction of measurement is fixed.
- Equation 2.18, Wigner rotation

$$W(\Lambda, p)^{\mu}{}_{\nu} = \left[L^{-1}(\Lambda p)\,\Lambda\,L(p)\right]^{\mu}{}_{\nu}$$

- Figure 1

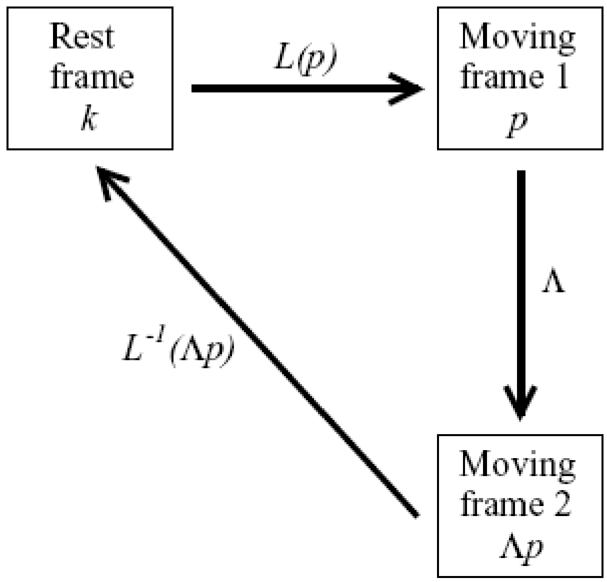

Figure 1: The Wigner rotation.

- Equation 2.20, spin transformation depends on the form of the momentum in general

$$U(\Lambda)\,|p,\sigma\rangle = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))\,|\Lambda p,\sigma'\rangle .$$

- Equation 2.21, under spatial rotations, the state |p, σ > transforms as in non-relativistic quantum mechanics

$$U(R)\,|p,\sigma\rangle = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(R)\,|Rp,\sigma'\rangle .$$

- Physical picture on p11 and figure 2, equations 2.40, 2.41.

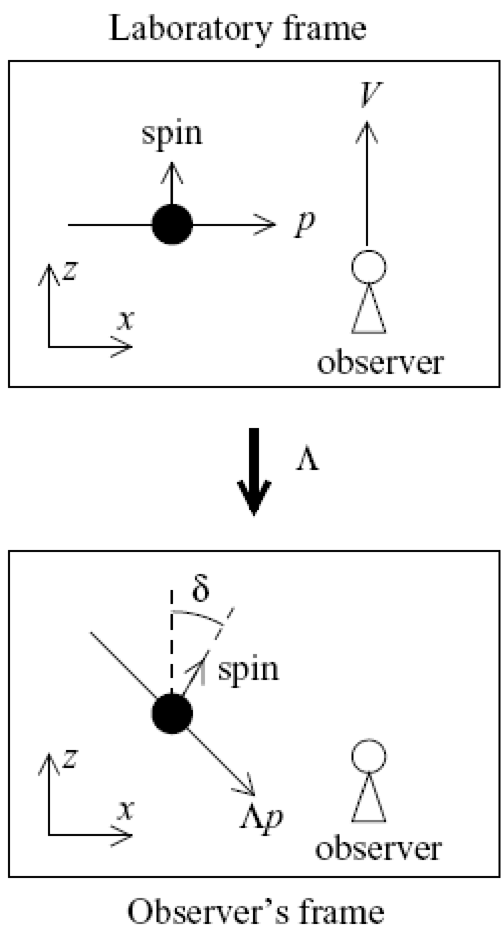

Figure 2: The orthogonal observer.

$$\vec{\Lambda p} = (Mc \sinh\xi, 0, -Mc\cosh\xi\sinh\chi),$$

$$\tan\delta_p = \left|\frac{(\Lambda p)^3}{(\Lambda p)^1}\right| = \frac{\sinh\chi}{\tanh\xi}.$$

  - The Lorentz transformation rotates the direction of the momentum eq 2.40
  - The spin is dragged by this rotation since the spin is coupled to the momentum in relativistic QM.
  - The spin rotation is <= the momentum rotation.
  - In the non-relativistic limit the spin angle does not rotate.

- The Wigner rotation can also be understood in terms of the unitary operator

We can also understand the Wigner rotation in terms of the unitary operator. It is the difference between $\Lambda L(p)^{\mu}{}_{\nu}$ and $L(\Lambda p)^{\mu}{}_{\nu}$ that gives rise to the Wigner rotation (2.18), as illustrated in Fig. 1. These are not equal, even though both of them bring the momentum k^{μ} to Λp^{μ}. In the limit of $\xi \to 0$ and $\chi \to 0$,

$$U(\Lambda)U(L(p)) = e^{-\frac{i}{\hbar}K^3\chi}e^{\frac{i}{\hbar}K^1\xi}, \tag{2.42}$$

$$U(L(\Lambda p)) \simeq e^{\frac{i}{\hbar}\left(-K^3\chi+K^1\xi\right)}, \tag{2.43}$$

where $\vec{K}$ is the boost operator (A.20). Using the Baker-Campbell-Hausdorff formula and the commutation relation $[K^1, K^3] = i\hbar J^2$, we obtain

$$U(\Lambda)U(L(p)) \simeq U(L(\Lambda p))\, e^{-\frac{i}{\hbar}J^2\frac{\xi\chi}{2}}. \tag{2.44}$$

This means that $U(\Lambda)\,|p,\sigma\rangle$ is different from $|\Lambda p,\sigma\rangle$ by the rotation of the spin about the y-axis, because

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= U(\Lambda)U(L(p))\,|k,\sigma\rangle^{\text{rest}} \\ &\simeq U(L(\Lambda p))\, e^{-\frac{i}{\hbar}J^2\frac{\xi\chi}{2}}\,|k,\sigma\rangle^{\text{rest}}, \end{aligned} \tag{2.45}$$

but

$$|\Lambda p,\sigma\rangle = U(L(\Lambda p))\,|k,\sigma\rangle^{\text{rest}}. \tag{2.46}$$

Massless Particle

- The massless case produces qualitatively similar results.
- Equation 3.22

$$W(\Lambda, p)^{\mu}{}_{\nu} = \left[L^{-1}(\Lambda p)\, \Lambda\, L(p)\right]^{\mu}{}_{\nu} \,.$$

- Equations 3.24 and 3.26, the plane of polarization is rotated by $\gamma(\Lambda, p)$ due to the Lorentz transformation.

$$U(\Lambda)\,|p, \sigma\rangle = e^{i\gamma(\Lambda,p)\sigma}\,|\Lambda p, \sigma\rangle,$$

$$U(\Lambda)\,|p; \zeta\rangle_{\pm} = |\Lambda p; \zeta + \gamma(\Lambda, p)\rangle_{\pm}.$$

EPR Correlation

- Non-relativistic EPR state in equation 4.1

$$|\phi\rangle = \frac{1}{\sqrt{2}}\Big[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \Big],$$

- Figure 5 shows the relativistic EPR experiment.

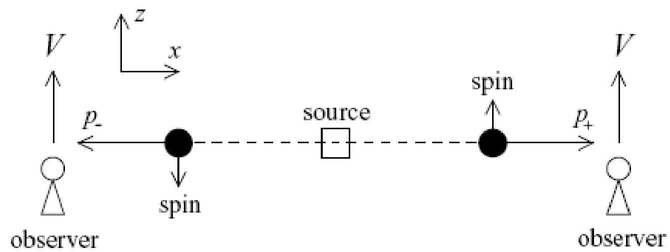

Figure 5: The relativistic EPR experiment in the laboratory frame.

- The spin-singlet state is mixed with the spin-triplet state by Lorentz transformation losing anti-correlation.

- Massive case
    - Equations 4.8, 4.9

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\,|p_+,\uparrow\rangle|p_-,\downarrow\rangle - |p_+,\downarrow\rangle|p_-,\uparrow\rangle\,\Big], \tag{4.8}$$

where

$$p^\mu_\pm = (Mc\,\cosh\xi, \pm Mc\,\sinh\xi, 0, 0). \tag{4.9}$$

- Figure 6 is in observers frame see equation 4.10

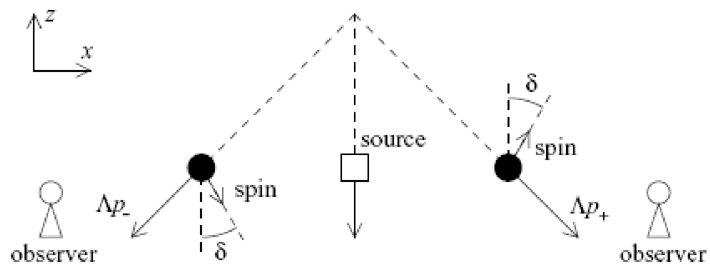

Figure 6: The relativistic EPR experiment in the observers' frame.

$$U(\Lambda)|\psi\rangle =$$
$$\frac{1}{\sqrt{2}}\Bigg[\cos\delta\Big(|\Lambda p_+,\uparrow\rangle|\Lambda p_-,\downarrow\rangle - |\Lambda p_+,\downarrow\rangle|\Lambda p_-,\uparrow\rangle\Big)$$
$$+\sin\delta\Big(|\Lambda p_+,\uparrow\rangle|\Lambda p_-,\uparrow\rangle + |\Lambda p_+,\downarrow\rangle|\Lambda p_-,\downarrow\rangle\Big)\Bigg], \tag{4.10}$$

- Spin-singlet state is mixed with the spin-triplet state
- Results are not always anti-correlated.

- Massless case
    - Relativistic EPR equation 4.12 and 4.13

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\,|p_+;\zeta\rangle_+|p_-;\zeta\rangle_- - |p_+;\zeta\rangle_-|p_-;\zeta\rangle_+\Big], \tag{4.12}$$

$$\begin{aligned} U(\Lambda)|\psi\rangle \;=\; & \frac{1}{\sqrt{2}}\Big[\,|\Lambda p_+;\zeta+\varepsilon\rangle_+|\Lambda p_-;\zeta-\varepsilon\rangle_- \\ & -|\Lambda p_+;\zeta+\varepsilon\rangle_-|\Lambda p_-;\zeta-\varepsilon\rangle_+\Big], \end{aligned} \tag{4.13}$$

- Thus the measurements of the polarization with respect to the same angle is not always anti-correlated

Bell's Inequality

- Massive case
  - Equation 5.10 is the relativistic Bell's Inequality

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}\,\cos^2\delta, \qquad (5.10)$$

  - An apparent decrease in the degree of violation of the Bell's Inequality results from the Lorentz transformation rotating the directions of the spin in a different manner.
  - Since the rotations are local transformations, they preserve the perfect anti-correlation in an appropriate direction
- Massless case
  - Equation 5.19 is Bell's Inequality for the massless case.

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}\cos 4\varepsilon, \qquad (5.19)$$