

Generalized second law of black hole thermodynamics and quantum information theory

A. Hosoya, A. Carlini, and T. Shimomura

Department of Physics, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152-8550, Japan

(Received 17 August 2000; published 9 April 2001)

We propose a quantum version of a gedanken experiment which supports the generalized second law of black hole thermodynamics. A quantum measurement of particles in the region outside of the event horizon decreases the entropy of the outside matter due to the entanglement of the inside and outside particle states. This decrease is compensated, however, by the increase in the detector entropy. If the detector is conditionally dropped into the black hole, depending on the experimental outcome, the decrease of the matter entropy is more than compensated for by the increase of the black hole entropy via the increase of the black hole mass, which is ultimately attributed to the work done by the measurement.

DOI: 10.1103/PhysRevD.63.104008

PACS number(s): 04.70.Dy, 03.67.—a

I. INTRODUCTION

The striking parallelism [1–4] between black hole physics and the laws of thermodynamics has attracted many physicists. However, it seems there remains some controversy concerning the question of to what extent the parallelism works for the second law. The generalized second law of black hole thermodynamics states that the sum of the Bekenstein-Hawking black hole entropy, which is a quarter of the area A of the event horizon, and the ordinary thermodynamical entropy S^m , i.e.,

$$S^{\text{tot}} = \frac{A}{4} + S^m, \quad (1)$$

never decreases.

One of the directions in the study of the generalized second law is based on a gedanken experiment suggested by Bekenstein [2,5] and discussed by many people. A box of mass E^b and entropy S^b is lowered by a string from infinity to a point near the horizon. The box is then thrown into the black hole. Apparently, the entropy contained in the box is gone for the outside region of the black hole. This decrease of entropy for the outside region of the black hole is compensated for by the increase of the Bekenstein-Hawking entropy of the black hole caused by the work done by the observer at the other end of the string located at infinity, as Unruh and Wald [6,7] showed by taking into account the buoyancy force of the thermal atmosphere of the quantum fields surrounding the black hole. In the present work, we pursue this direction of research by elucidating the role of the entanglement between the states of the inside and the outside regions of a black hole, and by applying a quantum information theoretical approach [8] to a similar gedanken experiment in the black hole spacetime. This is also motivated by the so called “holographic principle” approach [9–11], according to which the information S_V contained within any region V is upper bounded by the area A_B of its boundary ($S_V \leq A_B/4$).

We shall start with an elementary introduction of quantum information entropy, and then go over to the generalized second law of black hole thermodynamics.

II. ENTANGLEMENT ENTROPY AND ROLE OF THE DETECTOR IN THE SECOND LAW

Let us start by discussing the ordinary second law by using quantum information theory [8]. This will elucidate the role of the entanglement of the states for the inside and outside region of a black hole, and especially the important role of the detector in a measurement process.

Let the initial state be

$$|\psi\rangle = \sum_n \sqrt{c_n} |n\rangle_B |n\rangle_A |\Phi_0\rangle, \quad (2)$$

in a Schmidt decomposition form. Here $|n\rangle_A$ is Alice’s state and $|n\rangle_B$ is Bob’s state, which are entangled. We have also introduced a detector state, which is initially in the ground state $|\Phi_0\rangle$. Note that, without loss of generality, we can assume that the c ’s are ordered, i.e., that $c_0 \geq c_1 \geq \dots \geq 0$, and that the detector state is either $|\Phi_0\rangle$ or $|\Phi_1\rangle$.

Suppose further that we are completely ignorant about Bob’s states so that we have to trace over them to obtain a density matrix for Alice’s and the detector states, i.e.,

$$\rho = \text{Tr}_B |\psi\rangle\langle\psi| = \sum_n c_n \left| n \right\rangle_A \left\langle n \right| \left| \Phi_0 \right\rangle \left\langle \Phi_0 \right| \left\langle n \right|_A, \quad (3)$$

and the corresponding von Neumann entropy reads

$$S = -\text{Tr}(\rho \log \rho) = -\sum_n c_n \log c_n, \quad (4)$$

which is also called the entanglement entropy.

Now let us switch on the apparatus. In a typical situation, Alice and the detector states will undergo the unitary evolution U_{nm}^α ($\alpha=0,1$), i.e.,¹

¹Note that the detector outside the horizon of a static black hole can only locally measure the “outgoing” modes $|n\rangle_A$, since the “ingoing” $|n\rangle_B$ modes have support on a causally disconnected region, see, e.g., Ref. [12].

$$|\psi\rangle \rightarrow |\psi'\rangle = \sum_n \sqrt{c_n} |n\rangle_B \sum_{m,\alpha} U_{nm}^\alpha |m\rangle_A |\Phi_\alpha\rangle. \quad (5)$$

Then switch off the apparatus again. The measurement process involves a separation of the wave packet into macroscopically distant ones corresponding to the ‘‘collapse of the wave function’’ (e.g., imagine an experiment in the manner of Stern and Gerlach) for different values of α . For each α , the density matrix and the weighted mean entropy \bar{S}' are given by

$$\rho'^\alpha = \sum_{n,m,m'} \frac{c_n}{K_\alpha} U_{nm}^\alpha U_{nm'}^{*\alpha} |m\rangle_A |\Phi_\alpha\rangle \langle \Phi_\alpha| \langle m'|_A, \quad (6)$$

$$\bar{S}' = \sum_\alpha K_\alpha S'^\alpha; S'^\alpha = -\text{Tr}(\rho'^\alpha \log \rho'^\alpha),$$

where $K_\alpha = \sum_{m,n} c_n |U_{nm}^\alpha|^2$ is the probability to get the state $|\Phi_\alpha\rangle$ by measurement, and $\sum_\alpha K_\alpha = 1$. Moreover, we can show from the concavity of the entropy that $\bar{S}' \leq S(\sum_\alpha K_\alpha \rho'^\alpha) = S(\rho)$. Therefore the average entropy decreases due to the measurement process.

We then do a local unitary transformation T^α to Bob's state $|n\rangle_B$ knowing the outcome α of the experiment via classical communication, so that the resultant state is of the Schmidt form

$$|\psi''^\alpha\rangle = \sum_m \frac{1}{\sqrt{K_\alpha}} W_m^\alpha |m\rangle_B |m\rangle_A |\Phi_\alpha\rangle. \quad (7)$$

Here the unitary matrix T^α is arranged so that $\sum_n \sqrt{c_n} T_{ni}^\alpha U_{nm}^\alpha = W_m^\alpha \delta_{i,m}$ holds. For each α the density matrix is $\rho''^\alpha = [\sum_m |W_m^\alpha|^2 |m\rangle_A \langle m|_A] / K_\alpha$ and the corresponding conditional entropy is $S''^\alpha = \log K_\alpha - [\sum_m |W_m^\alpha|^2 \log |W_m^\alpha|^2] / K_\alpha$. Then the total entropy sum of the detector entropy

$$S^d = -\sum_\alpha K_\alpha \log K_\alpha, \quad (8)$$

and of the average entropy, $\bar{S}'' \equiv \sum_\alpha K_\alpha S''^\alpha$ ($\bar{S}'' = \bar{S}'$), can be easily shown to increase,

$$\bar{S}'' + S^d \geq S. \quad (9)$$

Therefore, the apparent violation of the ordinary second law due to the decrease in the average matter entropy is avoided by properly taking into account the role of the detector entropy.

One might note that our model [as for the decrease of the average entropy after measurement, Eqs. (2)–(6)] is a particular realization of the general framework of Ref. [13], where it is shown that by local operations which consist of unitary transformations of Alice's and the detector states, and via a POVM process, one can have the following transition: $\sum_n \sqrt{c_n} |n\rangle_B |n\rangle_A \rightarrow \sum_n \sqrt{c'_n} |n\rangle_B |n\rangle_A$, with $c'_0 \geq c_0$, $c'_0 + c'_1 \geq c_0 + c_1$, \dots , $c'_0 + c'_1 + \dots = c_0 + c_1 + \dots = 1$, i.e., where the $\{c'\}$'s ‘‘majorize’’ the $\{c\}$'s. In our case, c'_n

$= |W_n^\alpha|^2 \equiv \sum_m c_m |U_{nm}^\alpha|^2$, and the majorization implies that $-\sum_n c'_n \log c'_n \leq -\sum_n c_n \log c_n$ (from the concavity of the entropy).

III. BEKENSTEIN'S GEDANKEN EXPERIMENT REEXAMINED

As it is by now well known, the radiation state of an eternal black hole can be described (in a semiclassical approximation for the background geometry) by the Hartle-Hawking state:

$$|\psi_{HH}\rangle = \prod_n \sum_n \sqrt{c_n} |n\rangle_B |n\rangle_A, \quad (10)$$

where $c_n = \exp(-(n\omega/T_{\text{BH}})/Z)$, T_{BH} is the Hawking temperature of the black hole, Z is the partition function, $|n\rangle_B$ is the n -particle state for the region inside the event horizon, and $|n\rangle_A$ is that for the region outside the event horizon. Hereafter, to simplify the presentation, we will consider only a particular mode of the particle states with angular frequency ω . The state $|\psi_{HH}\rangle$ is an entangled state of Alice's and Bob's states (Alice and Bob can see only $|n_A\rangle$ and $|n_B\rangle$, respectively), which is naturally prepared in general relativity.

Imagine we do a local operation by observing Alice's states. Tracing over Bob's states, we see that the entanglement entropy decreases as we have shown in Sec. II. Of course, this does not necessarily imply the violation of the second law, since by taking into account the detector entropy the total entropy will still increase as we have seen before.

However, what happens if the detector carrying the entropy acquired by the measurement is dropped into the black hole? It seems that the loss in radiation entropy caused by the local operation has no way to be compensated by the detector entropy, which is now gone into the black hole. This is a quantum analogue of the ‘‘classical’’ gedanken experiment proposed by Bekenstein [2,5] and discussed by Unruh and Wald [6,7].

Before discussing our ‘‘quantum’’ gedanken experiment, we reexamine Bekenstein's ‘‘classical’’ gedanken experiment by exploiting a fundamental inequality of ordinary thermodynamics between the work done on a system and the change of the total Helmholtz free energy. Suppose that the black hole has reached thermal equilibrium with the surrounding radiation, the whole system being enclosed in a large cavity whose boundary is located far from the horizon so that its temperature equals the Hawking temperature T_{BH} . Then, the local temperature of the thermal radiation is given by Tolman's law as $\tilde{T} = T_{\text{BH}}/\chi$, where χ is the local redshift factor. In the gedanken experiment à la Bekenstein, the box containing some energy E^b and entropy S^b is initially located outside the cavity [A in Fig. 1(a)], then slowly lowered down by means of a string toward the black hole [B in Fig. 1(b)] and finally dropped into the black hole from some point near the horizon [Fig. 1(c)].

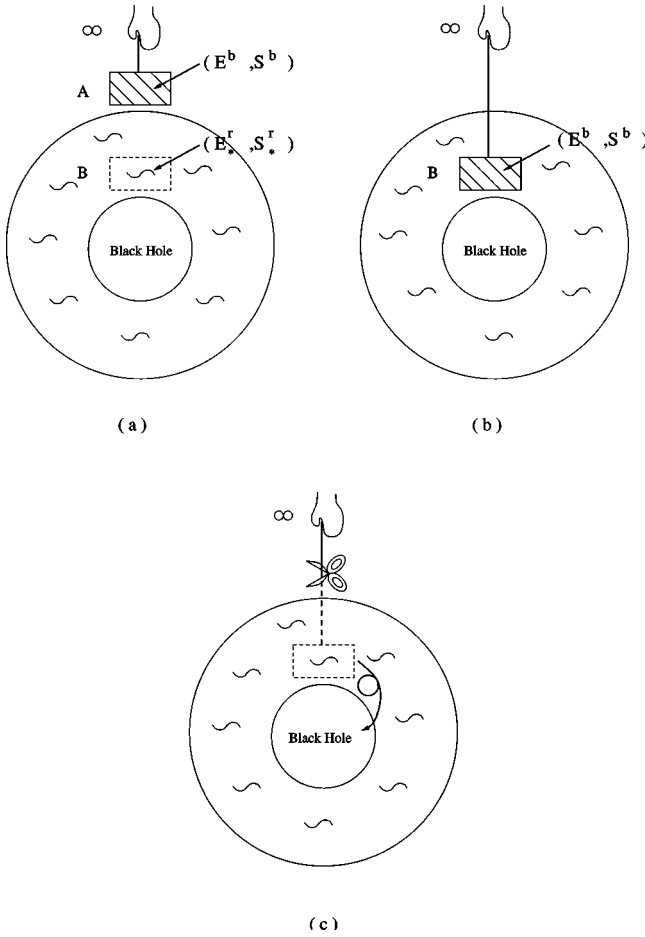


FIG. 1. The box is initially located outside the cavity and then lowered down toward the region near the horizon. Finally, we cut the string and the box is thrown into the black hole.

In this process, the change in the total entropy is

$$\Delta S^{\text{tot}} = \frac{\Delta M}{T_{\text{BH}}} + \Delta S^m, \quad (11)$$

where ΔM and $\Delta S^m = -S^b$ are the change in the black hole mass and that in matter entropy, respectively. Furthermore, the energy conservation law implies that

$$\Delta M = E^b + \Delta W, \quad (12)$$

where E^b is the initial energy of the box and ΔW is the work done by the agent at infinity in this process. Note that, since ΔM is the change in the Arnowitt-Deser-Misner (ADM) mass within the cavity, and ΔW is the work done by the agent at infinity to the whole system of the black hole and the thermal atmosphere.

Noting that the value of the temperature at the boundary of the cavity is fixed during this process, we can evaluate the work ΔW done to the system at infinity during this isothermal process by using the ordinary thermodynamics as [14]

$$\Delta W \geq F_f - F_i, \quad (13)$$

where F_f (F_i) is the Helmholtz free energy of the final (initial) stage observed from infinity.² The equality in Eq. (13) holds for a quasistatic process. Assuming that the lowering process does not disturb the bulk of the thermal radiation and neglecting the terms which cancel after the subtraction, each term on the right-hand side of the inequality Eq. (13) can be written as

$$F_f = (E_*^b - T_*^b S_*^b) \chi_*,$$

$$F_i = (E^b - T_\infty^b S^b) + (E_*^r - \tilde{T}_*^r S_*^r) \chi_*, \quad (14)$$

where the quantities E^r and S^r are the energy and entropy of the displaced thermal radiation, the index * means that the quantities are evaluated at the dropping point, and χ denotes the redshift factor. Assuming that the content of the box is completely shielded from the surrounding radiation by impenetrable walls, we can see that for the box the temperature $T_*^b \chi_* = T_\infty^b$, and the energy and entropy remain constant, $E_*^b = E^b$, $S_*^b = S^b$.

Therefore, putting Eqs. (11)–(14) together, we get

$$\Delta S^{\text{tot}} \geq \left(S_*^r - \frac{E_*^r}{T_{\text{BH}}} \chi_* \right) - \left(S^b - \frac{E^b}{T_{\text{BH}}} \chi_* \right), \quad (15)$$

where E_*^r , S_*^r , and χ_* are functions of the distance r of the box from the horizon. Equation (15) corresponds to Eqs. (16) and (A12) of Ref. [7], although we have not used model dependent arguments like that on the buoyancy force. The minimum of the function $\Delta S^{\text{tot}}(r)$ can be found by minimizing the change in the ADM mass ΔM , i.e., $d[\Delta S^{\text{tot}}(r)]/dr = [(E^b - E_*^r)/T_{\text{BH}}](d\chi_*/dr) = 0$, where we have used the first law of thermodynamics for the surrounding radiation. Using the terminology of Ref. [7], one can see that the minimum is realized when the box is dropped into the black hole from the floating point $E^b = E^r(r_{\text{fp}})$ where the gravity and buoyancy force on the box balance. Therefore, it is sufficient for the estimation of the lower bound on ΔS^{tot} to consider the case when we cut the string at the floating point r_{fp} to let the box freely fall into the black hole:

$$\Delta S^{\text{tot}} \geq (S_*^r - S^b)|_{r=r_{\text{fp}}} \geq 0, \quad (16)$$

where the last inequality follows from the fact that the maximum entropy is achieved by the thermal distribution for fixed values of volume and energy.

Therefore, by using only the physical properties of ordinary matter, we can show that the total entropy Eq. (11) always increases in this gedanken experiment within the first order approximation, i.e., when the changes in black hole mass and matter entropy are assumed to be small compared with the values of black hole mass and entropy themselves, and we can neglect the back reaction effects.

²Similar arguments can be easily generalized to the case of a charged, rotating (quasistationary) black hole background.

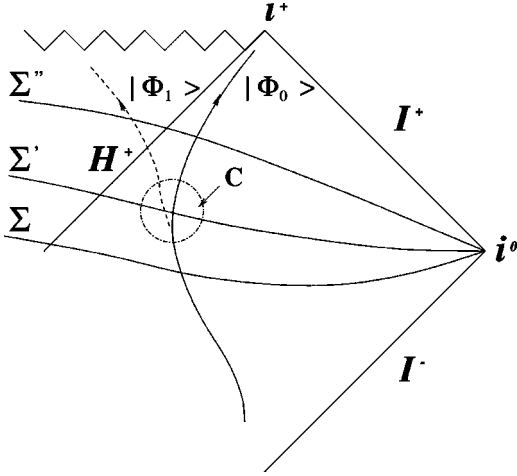


FIG. 2. We prepare the wave packet on the hypersurface Σ , and then perform a quantum measurement which splits the wave packet into macroscopically distant ones, according to the detector states $|\Phi_0\rangle$ and $|\Phi_1\rangle$ on the hypersurface Σ' . On the final hypersurface Σ'' only one of the packets, $|\Phi_0\rangle$, remains at the end of the string, while the other, $|\Phi_1\rangle$, is freely falling into the black hole.

We would like to emphasize that the generalized second law makes sense only when we consider the entropy of the outside region and can be shown through the ordinary second law of thermodynamics, assumed to hold locally in the outside region.

IV. THE GENERALIZED SECOND LAW AND QUANTUM INFORMATION

Now consider the ‘‘quantum’’ gedanken experiment which might seem to cause the violation of the generalized second law at first sight. Let us imagine a situation in which we perform a quantum measurement that splits the wave packet, initially prepared on the hypersurface Σ as in Eq. (2),³ according to the detector states $|\Phi_0\rangle$ and $|\Phi_1\rangle$ on the hypersurface Σ' . On the final hypersurface Σ'' , only one of the packets (e.g., $|\Phi_0\rangle$) remains at the end of the string, while the other one (e.g., $|\Phi_1\rangle$) is freely falling into the black hole (see Fig. 2). On the hypersurface Σ'' , the reduced density matrix $\rho^{''0}$ obtained by tracing over Bob’s states is given by Eq. (6) with $\alpha=0$.

By using Nielsen’s inequality, we have an apparent violation of the second law

$$S^{''0} \equiv -\text{Tr}_A(\rho^{''0} \log \rho^{''0}) \leq S = -\sum_n c_n \log c_n, \quad (17)$$

which is caused by the local operation involving the measurement.

³The generalized second law can be shown to hold even if the initial state of the detector on Σ is mixed.

However, the change in total entropy is

$$\Delta S^{\text{tot}} = \frac{\Delta W}{T_{\text{BH}}} + \Delta S^m \geq \frac{F_{\text{II}} - F_{\text{I}}}{T_{\text{BH}}} + \bar{S}'' - S^r, \quad (18)$$

where F_{II} (F_{I}) is the free energy of the detector and radiation after (before) the local operation. These quantities are evaluated in the region C in Fig. 2, and we obtain

$$F_{\text{II}} = [K_0(E_0'' - \tilde{T}S^{''0}) + K_1(E_1'' - \tilde{T}S^{''1})]\chi,$$

$$F_{\text{I}} = (E - \tilde{T}S^r)\chi, \quad (19)$$

where χ is the redshift factor in the small region C , E_α'' is the energy of the detector and radiation for a given α , and $\tilde{T}\chi = T_{\text{BH}}$ as before.

We are now close to the conclusion, since

$$\Delta S^{\text{tot}} \geq 0, \quad (20)$$

due to the energy conservation, $K_0 E_0'' + K_1 E_1'' = E$.

V. SUMMARY

A quantum version of Bekenstein’s ‘‘classical’’ gedanken experiment has been proposed. We performed a series of local operations which consist of certain unitary transformations for the outside and the detector states and a local measurement in the region outside the black hole. In general, this causes a decrease of the entanglement entropy by Nielsen’s inequality. However, as far as the detector remains outside, the total entropy including the detector one always increases. When the detector is conditionally dropped into the black hole depending on the experimental outcome, the decrease of the matter entropy, $\Delta S^m = -S^b$, caused by dropping the box is more than compensated by the increase of the black hole entropy, $\Delta S^{\text{BH}} = \Delta M/T_{\text{BH}}$, which ultimately arises from the work done by the measurement via the first law. Therefore, the generalized second law holds.

We here comment briefly on the work by Frolov and Page [15]. Although they proved the generalized second law for eternal black holes by comparing the initial vacuum state before the collapse of a star and the final entangled state, their situation is quite different from our gedanken experiment (e.g., nothing equivalent to the detector is thrown into the black hole). Finally, we would like to make the following remark. For the change of the total entropy in the gedanken experiment of Sec. III, Unruh and Wald have derived the same expression as Eq. (15) by using the buoyancy force exerted on the box by the black hole atmosphere. So far, several discussions about the practical effects of this force on the energetics of the process have been made [4]. In particular, Bekenstein recently argued [16] that the computation of the Unruh-Wald buoyancy force using the stress-energy

tensor is invalid for the case in which the size of the box is smaller than the typical wavelength of the accelerated radiation, and that his entropy bound $S^b/E^b \leq 2\pi R$ (R being the characteristic size of the box) is necessary in order for the generalized second law to be valid. In our derivation, we have not used model dependent arguments like the one on the buoyancy force, and this is in the spirit that the generalized second law should be a fundamental law of nature.

ACKNOWLEDGMENTS

A.H. was partially supported by the Ministry of Education, Science, Sports and Culture of Japan, under Grant No. 09640341. A.C. was supported by JISTEC, under STA Grant No. 199016; he also thanks the cosmology group at TITECH and the photonic group at CRL for their kind hospitality during this work.

-
- [1] J.M. Bardeen, B. Carter, and S. Hawking, *Commun. Math. Phys.* **31**, 161 (1973).
 - [2] J.D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
 - [3] S.W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
 - [4] R.M. Wald, "The Thermodynamics of Black Holes," gr-qc/9912119.
 - [5] J.D. Bekenstein, *Phys. Rev. D* **23**, 287 (1981).
 - [6] W.G. Unruh and R.M. Wald, *Phys. Rev. D* **25**, 942 (1982).
 - [7] W.G. Unruh and R.M. Wald, *Phys. Rev. D* **27**, 2271 (1983).
 - [8] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, Netherlands, 1995).
 - [9] G. 't Hooft, "Dimensional Reduction in Quantum Gravity," gr-qc/9310026.
 - [10] W. Fischler and L. Susskind, "Holography and Cosmology," hep-th/9806039.
 - [11] R. Bousso, *J. High Energy Phys.* **06**, 028 (1999).
 - [12] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982), p. 278.
 - [13] M.A. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999).
 - [14] L.D. Landau and E. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980), p. 61.
 - [15] V.P. Frolov and D.N. Page, *Phys. Rev. Lett.* **71**, 3902 (1993).
 - [16] J.D. Bekenstein, *Phys. Rev. D* **60**, 124010 (1999).